ON THE MOST PROBABLE VALUE OF THE LATITUDE, AND ITS THEORETICAL WEIGHT, FROM ENTANGLED OBSERVATIONS OCCURRING IN THE USE OF TALCOTT'S METHOD.

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Each observation of a pair of stars gives an equation of the well-known form,

$$\varphi = \tfrac{1}{2}(\delta_s + \delta_n) + \tfrac{1}{2}(M_s - M_n)R + \tfrac{1}{2}(l_s + l_n) + \tfrac{1}{2}(r_s - r_n) + \tfrac{1}{2}(\zeta_s + \zeta_n); \quad (1)$$

where the terms of the right-hand member depend respectively on the star-declinations, the micrometer readings, the level readings, the refractions, and the reductions to the meridian.

The entanglement consists in the use of the same star in more than one pair. One or more stars on one side of the zenith may be combined with one or more stars on the other side of the zenith. For the sake of uniformity in the reductions and of ready detection of errors in the star-places or instrumental readings, it is best to combine the stars in simple pairs. Unless then, with a given setting of the level, the number of stars observed on each side of the zenith is the same, some star, or stars, on one side must be combined, individually, with more than one star on the other side of the zenith.

Suppose the star whose declination is δ_A to be combined with the stars on the other side of the zenith, whose declinations are $\delta_1, \delta_2, \delta_3, \ldots$ Let these symbols denote the mean declinations of the stars.

Since the reductions to apparent place for the several dates do not come into consideration for the purpose of this paper, we may, for brevity, represent any functions of them whatever that may occur in our preliminary equations by the symbols $\Delta_A, \Delta_1, \Delta_2, \Delta_3, \ldots$ for the corresponding stars, as indicated by the suffixes.

Let

$I_A, I_1, I_2, I_3, \ldots$ = values, for the corresponding stars as indicated by the suffixes, of the right hand member of equation (1) exclusive of the first term,

$n_1, n_2, n_3, \ldots$ = number of observations of the pairs into which $\delta_1, \delta_2, \delta_3, \ldots$ enter,

ν = total number of nights on which observations were made of any pairs of the group,

μ = number of pairs in the group,

$m_1, m_2, m_3, \ldots$ = number of pairs of the group observed on the 1st, 2nd, 3rd, etc., nights,

$\varphi_1, \varphi_2, \varphi_3, \ldots$ = arithmetical means of the values of the latitude resulting from the individual observations of the 1st, 2nd, 3rd, etc., pairs,

φ_0 = most probable value of the latitude from the entangled group of pairs under consideration,

$p_1, p_2, p_3, \ldots$ = weights with which $\varphi_1, \varphi_2, \varphi_3, \ldots$ are to be combined to produce φ_0,

$P = p_1 + p_2 + p_3 + \ldots = \Sigma p$,

W_0 = theoretical weight of φ_0,

$v_1, v_2, v_3, \ldots$ = residual corrections of $\varphi_1, \varphi_2, \varphi_3 \ldots$ with reference to φ_0,

r = probable error of a latitude in general,

r_0 = probable error of φ_0,

r_δ = probable error of a star-declination,

r_I = probable error of a single observation of a single star with the zenith telescope = probable error of any one of the quantities I_A, I_1, I_2, I_3, etc.,

$$h = r_I^2 / r_\delta^2. \tag{2}$$

With a good instrument and a good observer we may assume $h = 2$, or $r_I^2 = 2r_\delta^2$. With adopted declinations depending on three or more good authorities, which condition can now be fulfilled generally, we may assume r_δ to have a common value, about $\pm$ 0''.50, for all the stars.

1. From the successive pairs of stars we have

$$\begin{aligned} \varphi_1 &= \tfrac{1}{2}(\delta_A + \delta_1) + \Delta_A + \Delta_1 + \frac{1}{2n_1}(\Sigma_1 I_A - \Sigma I_1), \\ \varphi_2 &= \tfrac{1}{2}(\delta_A + \delta_2) + \Delta_A + \Delta_2 + \frac{1}{2n_2}(\Sigma_2 I_A - \Sigma I_2), \end{aligned} \tag{3}$$

where the meaning of the signs of summation is quite evident; and, for the determination of φ_0, the equations

$$\begin{aligned} \varphi_0 - \varphi_1 &= v_1, \\ \varphi_0 - \varphi_2 &= v_2, \\ &\ldots\ldots \end{aligned}$$

to which proper weights $p_1, p_2, \ldots$ are to be assigned. Whatever these weights may be we shall have

$$\varphi_0 = \frac{\Sigma(p\varphi)}{P}. \tag{4}$$

Substituting in (4) the values of $\varphi_1, \varphi_2, \varphi_3, \ldots$ from (3), we have

$$\varphi_0 = \tfrac{1}{2}\delta_A + \Delta_A + \frac{1}{2P}\sum_{i=1}^{i=\mu}(p_i\delta_i) + \Sigma_i\Delta_i$$

$$+\frac{1}{2P}\left[\left(\Sigma_i'^{(1)}\frac{p_i}{n_i}\right)I_A^{(1)}+\left(\Sigma_i'^{(2)}\frac{p_i}{n_i}\right)I_A^{(2)}+\ .\ .\ .\ +\left(\Sigma_i'^{(\nu)}\frac{p_i}{n_i}\right)I_A^{(\nu)}\right]$$

$$-\frac{1}{2P}\left[\frac{p_i}{n_i}\Sigma_1'^{n_1}I_1+\frac{p_2}{n_2}\Sigma_1'^{n_2}I_2+\ .\ .\ .\ +\frac{p_\mu}{n_\mu}\Sigma_1'^{n_\mu}I_\mu\right]. \tag{5}$$

In the above equation the summations, except within the last brackets, are with reference to pairs of stars, the suffix i indicating the quantities that change from pair to pair. The term $\sum\limits_{i=1}^{i=\mu}{}'(p_i\delta_i)$ is the only one that necessarily involves all the pairs. The parenthetical exponents (1), (2), . . . , within the first brackets, indicate that the quantities to which they are attached pertain to the 1st, 2nd, etc., nights. Within the last brackets the summations are with reference to nights of observation, as in (3).

So far as the declinations and micrometer readings are concerned, each term of (5) is independent of every other term. Hence, proceeding to probable errors, we have

$$4r_0^2=\left(1+\frac{1}{P^2}\sum_{i=1}^{i=\mu}{}'p_i^2\right)r_\delta^2$$

$$+\frac{1}{P^2}\left[\left(\Sigma_i'^{(1)}\frac{p_i}{n_i}\right)^2+\left(\Sigma_i'^{(2)}\frac{p_i}{n_i}\right)^2+\ .\ .\ .\ +\left(\Sigma_i'^{(\nu)}\frac{p_i}{n_i}\right)^2\right]r_I^2$$

$$+\frac{1}{P^2}\left(\sum_{i=1}^{i=\mu}{}'\frac{p_i^2}{n_i}\right)r_I^2,$$

where the only quantities that necessarily involve all the pairs are $\sum\limits_{i=1}^{i=\mu}{}'p_i^2$ and $\sum\limits_{i=1}^{i=\mu}{}'p_i^2/n_i$.

2. We have now to determine p_1, p_2, p_3 . . . under the condition that the resulting value of r_0 shall be a minimum.* It is evident that the maximum value of r_0 is the probable error of the value of φ from that pair which has the smallest number of observations; that is, the value of r_0 corresponding to the system of weights where the weight of that pair $=1$ and the weight of every other pair $=0$. We shall determine the minimum, therefore, if we differentiate the expression for $4r_0^2$ with reference to $p_1, p_2, p_3,$ . . . respectively.

If, for brevity, we write equation (6) in the form

$$4r_0^2=r_\delta^2+\frac{U}{P^2},$$

*_On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations._ George Biddell Airy. §10, ¶64, and §11, ¶69.

and differentiate with reference to $p_1, p_2, p_3, \ldots$, we have a series of equations, μ in number, of the form

$$\frac{dU}{dp} = F,$$

for the determination of $p_1, p_2, p_3, \ldots$; where F is a certain function common to all the μ equations, since P is a symmetrical function of $p_1, p_2, p_3, \ldots$. Writing dU/dp in full, we have for the weight p_t of any particular pair observed on n_t nights, the equation

$$p_t r_\delta^2 + \left\{ \Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} + p_t \right\} \frac{r_I^2}{n_t} = \frac{F}{2};$$

in which the terms corresponding to any nights on which the given pair may not have been observed, are equal to zero. Substituting hr_δ^2 for r_I^2, from (2), we have μ equations as follows, for the determination of $p_1, p_2, p_3, \ldots$;

$$\frac{n_1 + h}{n_1} . p_1 + \frac{h}{n_1} \left(\Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} \right) = \frac{F}{2r_\delta^2} = F',$$

$$\frac{n_2 + h}{n_2} . p_2 + \frac{h}{n_2} \left(\Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} \right) = \frac{F}{2r_\delta^2} = F', \quad (7)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From the solution of the equations (7) we have

$$p_1 = k_1 F', \qquad p_2 = k_2 F', \qquad \ldots, \quad (8)$$

where $k_1, k_2, k_3, \ldots$ are functions of known quantities. But, since we are concerned only with relative weights within the group of pairs, we may write

$$p_1 = k_1, \qquad p_2 = k_2, \qquad \ldots. \quad (9)$$

3. A common case in practice is where certain pairs of the group are observed on each of ρ nights only, while the remaining pairs of the group are observed on each of ν nights, including the same ρ nights. Let $\lambda =$ the number of pairs observed on ρ nights only, and let their combination-weights be represented by $p_1, p_2, p_3, \ldots p_\lambda$. Then the weights of the $\mu - \lambda$ remaining pairs will be represented by $p_{\lambda+1}, p_{\lambda+2}, \ldots p_\mu$. Evidently

$$p_1 = p_2 = \ldots = p_\lambda,$$

and

$$p_{\lambda+1} = p_{\lambda+2} = \ldots = p_\mu,$$

Let

$$s = \sum_{i=1}^{i=\mu} \frac{p_i}{n_i} = \lambda . \frac{p_\lambda}{\rho} + (\mu - \lambda) . \frac{p_\mu}{\nu}.$$

From (7) we have

$$(\rho + h) p_\lambda + h\rho s \qquad = \rho F',$$

$$(\nu + h) p_\mu + h\left[\rho s + (\nu - \rho)\left(s - \lambda \frac{p_\lambda}{\rho}\right)\right] = \nu F'.$$

The second of these equations reduces to

$$(\nu + h) p_\mu + h\nu s - h\lambda p_\lambda \left(\frac{\nu - \rho}{\rho}\right) = \nu F'.$$

Eliminating F' between these two equations, we have

$$[\nu(\rho + h) + h\lambda(\nu - \rho)] p_\lambda - \rho(\nu + h) p_\mu = 0.$$

Whence
$$\frac{p_\mu}{p_\lambda} = \frac{\nu(\rho + h) + h\lambda(\nu - \rho)}{\rho(\nu + h)}. \qquad (10)$$

The disappearance of μ in this expression is to be noted.

4. Instead of combining the values of φ resulting from the separate pairs of stars and then determining the relative weights of pairs, we might have combined the values of φ resulting from the separate nights and then have determined, according to the same principles, the relative weights of nights. It may be of interest to compare the expression thus derived corresponding to (10). If we let p_ρ and p_ν = the weight of any one of the ρ and of the $\nu - \rho$ nights respectively, we shall find, after the necessary reductions,

$$\frac{p_\rho}{p_\nu} = \frac{h\mu(\mu - \lambda + 1) + \lambda(\nu - \rho)}{h(\mu - \lambda)(\mu + 1)}.$$

For a symmetrical group of observations, under this case, we have $\lambda = \nu - \rho$, $\mu = \nu$, and, if we assume $h = 1$, the expressions for p_μ/p_λ and p_ρ/p_ν should become identical. We shall find

$$\frac{p_\mu}{p_\lambda} = \frac{\mu}{\mu - \lambda} - \frac{\lambda}{\mu + 1} = \frac{p_\rho}{p_\nu} = \frac{\nu}{\rho} - \frac{\nu - \rho}{\nu + 1}.$$

5. If we put

$$S = \left(\Sigma_i^{(1)} \frac{p_i}{n_i}\right)^2 + \left(\Sigma_i^{(2)} \frac{p_i}{n_i}\right)^2 + \ldots + \left(\Sigma_i^{(\nu)} \frac{p_i}{n_i}\right)^2 + \sum_{i=1}^{i=\mu} \left(\frac{p_i^2}{n_i}\right),$$

and substitute hr_δ^2 for r_I^2, equation (6) becomes

$$4r_0^2 = \left(1 + \frac{1}{P^2} \sum_{i=1}^{i=\mu} p_i^2 + \frac{h}{P^2} S\right) r_\delta^2. \qquad (11)$$

Then we have
$$W_0 = \frac{1}{r_0^2} = \frac{4P^2}{P^2 + \sum_{i=1}^{i=\mu} p_i^2 + hS} \cdot \frac{1}{r_\delta^2}. \qquad (12)$$

If we wish to give the weight unity to a single observation of a single independent pair, we must assume $r_\delta^2 = \frac{2}{3}$; whence by (12)
$$W_0 = \frac{6P^2}{P^2 + \sum_{i=1}^{i=\mu} p_i^2 + hS}. \qquad (13)$$

If all the pairs are observed on each of all the nights, evidently $p_1 = p_2 = p_3 = \ldots$, and (12) and (13) become respectively
$$W_0 = \frac{4\mu\nu}{(\mu + 1)(\nu + h)} \cdot \frac{1}{r_\delta^2}, * \qquad (14).$$

and
$$W_0 = \frac{6\mu\nu}{(\mu + 1)(\nu + h)}. \qquad (15$$

6. At some stations it will happen that a certain star is so near the zenith that it may be observed in both positions of the instrument, and forms a pair by itself. It may be well to present the principal formulæ above in the shape which they assume for this case. From (1) we have for a single such pair,
$$r^2 = r_\delta^2 + \frac{1}{2n} \cdot r_I^2 = \frac{2n + h}{2n} \cdot r_\delta^2;$$

and
$$r^2 = \frac{2n + h}{3n}, \quad \text{if} \quad r_\delta^2 = \frac{2}{3}.$$

If we suppose the stars whose declinations are denoted by δ_A and δ_1 to be identical, we have corresponding respectively to (11), (12), (13), (14), and (15) the following:—
$$4r_0^2 = \left[\left(\frac{P + p_1}{P} \right)^2 + \frac{1}{P^2} \sum_{i=2}^{i=\mu} p_i^2 + \frac{h}{P^2} S \right] r_\delta^2, \qquad (11')$$
$$W_0 = \frac{4P^2}{(P + p_1)^2 + \sum_{i=2}^{i=\mu} p_i^2 + hS} \cdot \frac{1}{r_\delta^2}, \qquad (12')$$
$$W_0 = \frac{6P^2}{(P + p_1)^2 + \sum_{i=2}^{i=\mu} p_i^2 + hS}, \qquad (13')$$

*Sir George B. Airy, in discussing a similar problem, that of determining the Weights to be given to the Separate Results for Terrestrial Longitudes, determined by the Observation of Transits of the Moon and Fixed Stars, has derived a formula which is the equivalent of (14). (Memoirs of the Royal Astronomical Society. Vol. XIX. p. 224).

$$W_0 = \frac{4\mu\nu}{(\mu + 1)(\nu + h) + 2\nu} \cdot \frac{1}{r_\delta^2}, \tag{14'}$$

$$W_0 = \frac{6\mu\nu}{(\mu + 1)(\nu + h) + 2\nu}. \tag{15'}$$

7. From (15) we have the following:—

Table of Weights when All the Pairs are Observed on Each of All the Nights.

$\mu =$ No. of Pairs.	$\nu =$ No. of Nights.				
	1	2	3	4	5
1	1.00	1.50	1.80	2.00	2.14
2	1.33	2.00	2.40	2.66	2.86
3	1.50	2.25	2.70	3.00	3.22
4	1.60	2.40	2.88	3.20	3.43
5	1.67	2.50	3.00	3.34	3.56

It is evident from the above table that in this case the weight of the value of the latitude from the group is increased by observations on additional nights rather than by observations on additional pairs; which is contrary to the usual case, that of independent pairs.

8. In practice the rigorous method of determining the values of p_1, p_2, p_3, . . . by (9), might sometimes occasion an undue amount of labor. It is not important oftentimes to determine theoretical weights rigorously, since, in general, a considerable change in the independent variable makes an inconsiderable change in the value of the function at or near a maximum or minimum. Sir George B. Airy shows, in his work on the Theory of Errors, already referred to by foot-note, that, given two independent measures whose combination-weights are as 4 : 1, any combination-weights from 2 : 1 up to 16 : 1 may be used without increasing the probable error of the result more than one-fifteenth part of itself. Still, a good computer would wish to use the best weights available, exercising his judgment as to the relative importance of a certain theoretical accuracy in the result and the labor necessary to attain theoretically that accuracy.

Sir George B. Airy, in his memoir referred to by foot-note under (14), after deriving, for a particular example, a set of equations corresponding to (9), recommends in general the use of certain approximate weights. His approximate

weight of any one observation, for the kind of observations we are considering, is the weight which that observation would have if it were assigned its equal proportionate part of the weight which the final value of the latitude from the entire group of observations would have, if, first, the total number of pairs observed were the same as the number of pairs on the night in question; and if, second, the total number of nights were the same as the number of nights on which the pair in question was observed; and if, third, all the pairs were observed on each of all the nights. This approximate weight is found, therefore, from (14) by dividing by $\mu\nu$, the number of observations, and substituting in the denominator for μ and ν, the corresponding m and n as described above. This gives

$$p = \frac{c}{(m+1)(n+h)}, \tag{16}$$

where c is an arbitrary constant determining the scale of the weights, and p is the weight of a single observation of a pair of stars.

9. For the case under which (10) was derived, a comparison is readily made of the ratio of the approximate weights from (16), with the ratio of the true weights from (10), and thus a better idea can be gained of their relations in general. From (16), disregarding the constant in the numerator, for this case we have

$$p_\lambda = \frac{\rho}{(\mu+1)(\rho+h)},$$

$$p_\mu = \frac{\rho}{(\mu+1)(\nu+h)} + \frac{\nu-\rho}{(\mu-\lambda+1)(\nu+h)}.$$

Whence
$$\frac{p_\mu}{p_\lambda} = \frac{\rho+h}{\nu+h}\left[1 + \frac{(\nu-\rho)(\mu+1)}{(\mu-\lambda+1)\rho}\right]. \tag{17}$$

If we let f_1 and f_2 equal the values of p_μ / p_λ from (10) and (17), respectively, we have

$$f_1 = \frac{\nu(\rho+h)}{\rho(\nu+h)} + \frac{h\lambda(\nu-\rho)}{\rho(\nu+h)},$$

$$f_2 = \frac{\nu(\rho+h)}{\rho(\nu+h)} \cdot \frac{\mu - \frac{\rho}{\nu}\lambda + 1}{\mu-\lambda+1};$$

or, transforming these expressions,

$$f_1 = \frac{\nu}{\nu+h} \cdot \frac{\rho+h}{\rho} \cdot \left[1 + \frac{\lambda(\nu-\rho)}{\nu} \cdot \frac{h}{\rho+h}\right], \tag{18}$$

$$f_2 = \frac{\nu}{\nu+h} \cdot \frac{\rho+h}{\rho} \cdot \left[1 + \frac{\lambda(\nu-\rho)}{\nu} \cdot \frac{1}{\mu-\lambda+1}\right]. \tag{19}$$

It is evident from these expressions that both f_1 and f_2 increase continually

as λ increases, and diminish continually as ρ increases. Hence each attains its greatest value when we have $\lambda = \mu - 1, \rho = 1$. For $\rho = \nu$ each becomes unity.

Subtracting (19) from (18) and reducing, we have, for the correction to the approximate ratio f_2,

$$f_1 - f_2 = \frac{1}{\nu + h} \cdot \frac{\nu - \rho}{\rho} \cdot \lambda \cdot \frac{h(\mu - \lambda) - \rho}{\mu - \lambda + 1}. \tag{20}$$

From this expression we have, evidently,

$$f_1 - f_2 \gtreqless 0 \quad \text{according as} \quad \rho \lesseqgtr h(\mu - \lambda);$$

and that $f_1 - f_2$ diminishes continually, in algebraic magnitude, as ρ increases. If we differentiate the factor $\lambda \cdot \frac{h(\mu - \lambda) - \rho}{\mu - \lambda + 1}$ with reference to λ, and put the result equal to zero, we find the values of λ that give a maximum or minimum of $f_1 - f_2$. The result, after reducing, is a quadratic equation; whence

$$\lambda = (\mu + 1) \pm \sqrt{\left[(\mu + 1) \cdot \frac{\rho + h}{h}\right]}.$$

It is evident from (20) that $f_1 - f_2$, for any given value of ρ, passes from positive to negative values as λ is increasing. Therefore the smaller value of λ above corresponds to the maximum of $f_1 - f_2$. Since λ cannot be greater than μ, the larger value of λ above, corresponding to the true minimum of $f_1 - f_2$, is inadmissible; but we have a least value of $f_1 - f_2$, algebraically, when $\lambda = \mu - 1$; although for $\lambda = \mu$, when also ρ must equal ν, we have $f_1 - f_2 = 0$. Hence we have the maximum value of $f_1 - f_2$ when $\lambda = (\mu + 1) - \sqrt{\left[(\mu + 1)\frac{\rho + h}{h}\right]}$, $\rho = 1$, and the least value when $\lambda = \mu - 1, \rho = \nu - 1$.

To compare numerical results, let us suppose the simple example where $\mu = 5, \nu = 5, h = 2$. By the above we have the maximum of $f_1 - f_2$ when $\lambda = 3, \rho = 1$. Substituting these numerical values in (20), we have $f_1 - f_2 = \frac{12}{7}$. Substituting also in (18) and (19), we have $f_1 = \frac{39}{7}, f_2 = \frac{27}{7}$; which are consistent with the value of $f_1 - f_2$. Hence f_2 is in error here by nearly one-third of the true value of the ratio of the weights. We have the least value of $f_1 - f_2$ when $\lambda = 4, \rho = 4$. Again substituting in (20), $f_1 - f_2 = -\frac{1}{7}$. The corresponding values of f_1 and f_2, from (18) and (19) are $\frac{19}{14}$ and $\frac{21}{14}$; so that here f_2 is in error only about one-tenth part. For $\mu = 23$, a higher number than would occur, of course, in latitude work, and $\nu = 5$, we should find for the maximum, $f_1 - f_2 = \frac{108}{7}$, while $f_1 = \frac{159}{7}$; whence f_2 would be in error by $\frac{108}{159}$ or 0.68 times the true value.

The numerical data, at least in the case of observations for latitude by Talcott's method, are so few and simple in any one example, it would seem more satisfactory to make a thorough solution for the weights, employing equation

(10) for the special case under which it was derived, than to depend on any system of approximate weights.

10. Mr. C. A. Schott* and Prof. T. H. Safford† have discussed, to some extent, the question of the final weight in the case of entangled observations for latitude by Talcott's method. Both writers consider the general case where N stars north of the zenith are observed with S stars south of the zenith with only one reversal of the instrument. Each, disregarding the error of observation, that is, taking into account the entanglement by declinations only, gives the readily derived expression $\frac{2NS}{N+S}$ as the factor by which the ordinary weight of the group, considered as one pair, is to be multiplied, before combining with values of the latitude from other pairs or groups of stars. In the case of such groups as we have under consideration, either N or S is unity, and the factor is of the form $\frac{2S}{1+S}$. Professor Safford gives a table of weights which, reduced to accord with the assumption of the weight unity for a single observation on a single independent pair, is as follows:

Number of Stars $= N+S$.	Number of observations of group.	Weight of mean value of latitude from group.
2	1	1.00
	2	1.50
	3 to 6	2.00
3		2.67
4	3 to 6	4.00
5		4.80

For a less number of observations than 3, the last three weights of the table are to be diminished in the proportion of the first three. It will be seen that the weights of the above table rise to a value exceeding one and one-half times the corresponding weights of the table computed by (15).

11. In computing the probable error of the final value of the latitude from

*Report of the Superintendent of the U. S. Coast and Geodetic Survey for 1880. Appendix No. 14, p. 255.

†Report of the Secretary of War 1878-9. Report of Capt. George M. Wheeler, Corps of Engineers, U. S. A. Surveys West of the One Hundredth Meridian, pp. 1987, 1995.

all the independent pairs and groups observed, by the formula

$$r = 0.6745 \sqrt{\left[\frac{\Sigma(pvv)}{(m-1)\,\Sigma p}\right]},$$

a group of entangled pairs should be represented by a single weight, $p = W_0$, and a single residual, v, the difference between φ_0 and the general mean from all the independent pairs. But if, in computing the final value of the latitude, it is desired for the sake of uniformity, to enter the values of the latitude from the individual pairs of the group, the arithmetical means of the individual observations on the several pairs may be multiplied respectively by the combination-weights,

$$p_1' = W_0 / P . p_1, \quad p_2' = W_0 / P . p_2, \quad \ldots ;$$

since

$$\sum_{i=1}^{i=\mu} p_i' = W_0 \quad \text{and} \quad \sum_{i=1}^{i=\mu} (p_i'\varphi_i) = W_0\varphi_0.$$

12. In some cases the same star will enter into more than one-group of entangled pairs. The value of the latitude is then to be computed from each group, by (4), and the values from the different groups combined by a process similar to that already employed in combining the individual pairs of a single group; as illustrated in the example given below.

13. The following observations, whose reduction will illustrate the application of the formulæ above, are taken from the Report upon United States Geographical Surveys West of the One Hundredth Meridian, in charge of First Lieutenant Geo. M. Wheeler, Corps of Engineers, U. S. Army; Vol. II. pp. 68–9. This example is presented not as one of a kind that frequently occur, or that would reward adequately the labor necessary for a thorough reduction, but as showing to an interesting extent the application of the above principles; which are applicable also to other kinds of observations than those for latitude.

Station: Beaver, Utah. Latitude $= + 38° 16'$.

Designation of Pairs.	Stars of Pair.		Observed Values of Latitude.		Mean $= \phi$.
	South.	North.	1872. August 23.	1872. August 24.	
(1)	B. A. C. 7733	B. A. C. 7754	25″.3	″ ——	25″.3
I	7807	7754	——	23.5	23.5
II	7807	7778	22.2	22.9	22.55
III	7807	7782	23.0	22.6	22.80
IV	7807	7779	——	23.2	23.2

We shall assign the weight 1 to a single observation of a single independent pair, in accordance with the assumptions $h = 2, r_\delta^2 = \frac{2}{3}$.

If we treat the pairs as independent and weight them by the ordinary formula $p = \frac{2n}{nr_\delta^2 + r_I^2}$, we have $\varphi_0 = 20'' + \frac{20''.03}{6} = 23''.34$; and the weight of φ_0 will be 6.00, the sum of the individual weights.

Since, in the example, the number of observations on any one of the pairs is so small, the weights of the table on p. 181 would approximate closely to the true theoretical weight. If we assume $n = 1\frac{1}{2}$ as a sort of average of the number of observations on each pair of stars, we have, from the system on which that table rests, 3.43 as the weight of the final value of the latitude.

We have here the pair (1) entangled with the group I, II, III, IV; so that we must first determine the most probable value of the latitude from the group, and then combine it suitably with the value from pair (1). For the group, which evidently comes under the case for which (10) was derived, we have $\mu = 4$, $\nu = 2$, $\lambda = 2$, $\rho = 1$, and we will assume throughout $h = 2$. By (10) we have

$$\frac{p_\mu}{p_\lambda} = \frac{2 \times 3 + 2 \times 2}{4} = \frac{5}{2}.$$

We have the following as the computation of φ_0 and W_0:

Pair.	ϕ	p	$p\phi$	p^2	p^2/n	p/n	
						Aug. 23.	Aug. 24.
I	23''.5	2	47''.0	4	4	—	2
II	22.55	5	112.8	25	$\frac{25}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
III	22.80	5	114.0	25	$\frac{25}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
IV	23.2	2	46.4	4	4	—	2
Sums		14	320.2	58	33	5	9

$$P^2 = 196\,. \left(\Sigma_i^{(1)} \frac{p_i}{n_i}\right)^2 = 25\,. \left(\Sigma_i^{(2)} \frac{p_i}{n_i}\right)^2 = 81,$$

$$S = 25 + 81 + 33 = 139,$$

and, by (15), $$\varphi_0 = \frac{320.2}{14} = 22''.87,$$

$$W_0 = \frac{6 \times 196}{196 + 58 + 2 \times 139} = 2.21.$$

Next let

φ' = the value of the latitude from pair (1),

φ_0' = the most probable value of the latitude from all the pairs of the example,

W', W'' = the combination-weights of φ', φ_0 required to produce φ_0',

r_0' = the probable error of φ_0',

W_0' = the theoretical weight of φ_0',

B, A, 1, 2, 3, 4 denote that the symbols to which they are attached pertain to the stars B. A. C. 7733, 7807, and the remaining stars in the order in which they occur in the example.

From (3) we have

$$\varphi' = \tfrac{1}{2}\left[\delta_B + \varDelta_B + \delta_1 + \varDelta_1 + (I_B - I_1)\right].$$

From (5) we have, bearing in mind that we represent in general by the symbol $\varDelta$ any function of the reduction to apparent place that may occur,

$$\varphi_0 = \tfrac{1}{2}\left[\delta_A + \varDelta_A + \frac{p_1}{P}\delta_1 + \varDelta_1 + \frac{1}{P}\sum_{i=2}^{i=\mu}(p_i\delta_i)\right] + \tfrac{1}{2}T;$$

where $\frac{1}{2}T$ represents all that is left of the right-hand member of (5) after taking out the five terms enclosed in the brackets of this last equation.

Substituting these values of φ' and φ_0 in the equation $\varphi_0' = \dfrac{W'\varphi' + W''\varphi_0}{W' + W''}$, we have

$$\varphi_0' = \frac{1}{2(W' + W'')}\left[W'\delta_B + W''\delta_A + \left(W' + \frac{p_1}{P}W''\right)\delta_1 + \frac{1}{P}W''\sum_{i=2}^{i=\mu}(p_i\delta_i)\right.$$
$$\left. + \varDelta_B + \varDelta_A + \varDelta_1\right] + \frac{1}{2(W' + W'')}\left[W'(I_B - I_1) + W''T\right].$$

From this we derive the equation for $r_0'^2$ just as (6) was derived from (5). Referring to the value of S, however, preceding (11), it appears that the square of the probable error of T is $\frac{1}{P^2}Sr_I^2$. Letting $r_I^2 = hr_\delta^2$ and combining terms with reference to W' and W'', we have

$$4r_0'^2 = \frac{1}{(W' + W'')^2}\left[2\,W'^2(1 + h) + \frac{2p_1}{P}W'W''\right.$$
$$\left. + W''^2\left(1 + \frac{p_1^2}{P^2} + \frac{1}{P^2}\sum_{i=2}^{i=\mu}p_i^2 + \frac{h}{P^2}S\right)\right]r_\delta^2.$$

The parenthesis after W''^2 reduces to $\frac{1}{P^2}\left(P^2 + \sum_{i=1}^{i=\mu}p_i^2 + hS\right)$, which, referring to (13), is the same as $6/W_0$, assuming $r_\delta^2 = \frac{2}{3}$. Hence we have

$$4r_0'^2 = \frac{1}{(W' + W'')^2}\left[2(1 + h)\,W'^2 + \frac{2p_1}{P}W'W'' + \frac{6}{W_0}W''^2\right]r_\delta^2 \quad (21)$$

From this expression we have to determine the values of W' and W'' that reduce $r_0'^2$ to a minimum; just as (7) was derived from (6). Differentiating, we have

$$4(1 + h)\,W' + 2\frac{p_1}{P}\,W'' = F',$$

$$\frac{2p_1}{P}\,W' + \frac{12}{W_0}\,W'' = F'.$$

For computing the numerical values of the coefficients of these equations, we have $h = 2$, and from the reduction of the I, II, III, IV above, $p_1 = 2$, $P = 14$, $W_0 = 2.21$. Solving the equations we obtain $W' / W'' = 0.44$; or $W' = 0.44$, $W'' = 1$. Hence

$$\varphi_0' = 20.'' + \frac{0.44 \times 5''.3 + 2''.87}{1.44} = 23''.61;$$

and from (21), assuming $r_\delta^2 = \frac{2}{3}$, and substituting for the other letters the numbers given above, we have $W_0' = 1 / r_0'^2 = 3.11$ instead of 6.00 as by the ordinary formula. Therefore, as the final result, we have: latitude $= + 38°16'\,23''.61$ with weight 3.11.

VENABLE'S MODERN GEOMETRY.*

This little manual, prepared as an appendix to Professor Venable's edition of Legendre's Elements of Geometry, is a worthy addition to that excellent text-book. The same judicious arrangement, care in selection of material, accuracy, and completeness of presentation characterize both.

The subjects treated are, in order: Transversals, and some of their applications to the geometry of position, after the methods of Carnot; anharmonic ratios, and the theorems of Pascal and Brianchon for the circle, with applications; harmonic rows and pencils, and the harmonic properties of the complete quadrilateral; poles and polars in the circle; reciprocal polars and the law of duality; radical axes; and axes and centres of similitude.

The treatment is rather metrical than descriptive, so that the learner would get from this manual a quite inadequate conception of the genuine and characteristic methods of the Modern Geometry as expounded by Steiner, von Staudt, Reye, Cremona, and their disciples; or even by Chasles. But in some respects, this alliance with the metrical methods of the Euclidean geometry is to be pre-

* Introduction to Modern Geometry. By Charles S. Venable, LL. D., Professor of Mathematics in the University of Virginia. University Publishing Company: New York, 1887.